Announcements

1) Amanda's Mentoring Hours.

MF 3-4:30 CB 2090

Tu 6-7 CB 2070

2) HW due date extended to Tuesday of next week.

Proposition (characterization of sup/inf) Let $S \subseteq \mathbb{R}$.

Then $\beta \in \mathbb{R}$ is the least upper bound of S if and only if β is an upper bound of S and for every $\varepsilon > 0$, $\exists\ x \in S$ with $\boxed{\beta - \varepsilon < x.}$

Proof $\Longrightarrow$ Suppose $\beta = \sup(S)$

Want to show: $\forall\ \varepsilon > 0$, $\exists\ x \in S$ with $\beta - \varepsilon < x$.

Choose $\varepsilon > 0$.

Want to find $x \in S$,

$\beta < x + \varepsilon$.

Let $x = \beta - \frac{\varepsilon}{2}$.

$x < \beta$, so since $\beta = \sup(S)$,

x cannot be an upper bound for the set S. Therefore,

$\exists$ $y \in S$, $x \leq y$ Then

$$y + \varepsilon \geq x + \varepsilon = \beta - \frac{\varepsilon}{2} + \varepsilon$$

$$= \beta + \frac{\varepsilon}{2} > \beta$$

Done with that implication.

$\Leftarrow$ Suppose, $\forall\ \varepsilon > 0$, $\exists\ x \in S$, $\beta < x + \varepsilon$. We also suppose β is an upper bound. By contradiction, suppose an upper bound γ of S, $\gamma < \beta$.

Let $\boxed{\varepsilon = \frac{\beta - \gamma}{2}}$. Then by assumption, $\exists\ x \in S$,

$$\beta < x + \varepsilon = x + \frac{\beta}{2} - \frac{\gamma}{2}$$

$$\frac{\beta}{2} < x - \frac{\gamma}{2}$$

Then $x > \frac{\beta}{2} + \frac{\gamma}{2}$

$$> \frac{\gamma}{2} + \frac{\gamma}{2} = \gamma$$

Since we assumed $\gamma < \beta$.

As $x \in S$, this shows γ is not an upper bound of S, contradiction. Therefore

$\gamma \geq \beta \Rightarrow \beta = \sup(S)$. $\square$

Consequences of Completeness

Theorem: (Nested Interval Property)

Let $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$, $x_n < y_n \ \forall n \in \mathbb{N}$. If

$$[x_{n+1}, y_{n+1}] \subseteq [x_n, y_n] \ \forall n \in \mathbb{N},$$

then $\bigcap_{n=1}^{\infty} [x_n, y_n] \neq \emptyset$

Proof If we fix an $n \in \mathbb{N}$, then $\forall k \in \mathbb{N}$,

$$x_n \le y_k .$$

Let $x = \sup\{x_n \mid n \in \mathbb{N}\}$.

By definition, $x \ge x_n \ \forall n \in \mathbb{N}$.

Since each y_k, $k \in \mathbb{N}$, is an upper bound for $\{x_n \mid n \in \mathbb{N}\}$, we have $x \le y_k \ \forall k \in \mathbb{N}$.

This implies $x_n \le x \le y_n$ $\forall n \in \mathbb{N}$, so $x \in [x_n, y_n]$.

This shows $x \in \bigcap_{n=1}^{\infty} [x_n, y_n]$

Therefore, $\bigcap_{n=1}^{\infty} [x_n, y_n] \neq \emptyset$. $\square$

Question: Why doesn't this work for open intervals?

What about half-open?

Examples: (open intervals)

$$I_n = \left(0, \frac{1}{n}\right)$$

$$\bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right) = \emptyset \quad \text{why?}$$

The next theorem!

Same proof works with

$$I_n = \left(0, \frac{1}{n}\right]$$

Theorem: (archimedean property)

$\forall x \in \mathbb{R}, \exists n \in \mathbb{N},$

$n > x$.

proof: If $x < 1$, choose $n = 1$.

Suppose $x \geq 1$. Argue by contradiction. If there is no $n \in \mathbb{N}$ with $n > x$, Then x is an upper bound for $\mathbb{N}$.

By the completeness axiom, $\mathbb{N}$ would then have a least upper bound y.

By our characterization of suprema, with $\varepsilon = 1$,

$\exists\, n \in \mathbb{N},\ y < n+1$

Contradiction since $n+1 \in \mathbb{N}$

Therefore, $\forall\, x \in \mathbb{R},\ \exists\, n \in \mathbb{N},$

$n > x$ $\square$